

ON A DISCRETE VERSION OF LENGTH METRICS

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ABSTRACT. Let (X, d) be a metric space. We study a metric d_0 on X naturally derived from d . If (X, d) is complete and locally compact, or if it is complete and $(d_0)_0 = d_0$, then d_0 coincides with the length metric induced by d . Counterexamples are constructed when any of the hypotheses is absent. The behavior of the iterates of d_0 (the metrics d_0^n recursively defined as $(d_0^{n-1})_0$) is also considered.

0. INTRODUCTION

Most of the definitions and many of the unproved facts stated below can be found in both [1] and [2], and the notation used here largely coincides with that of [2].

Let (X, d) be a metric space. Let $P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$ denote a partition of $[0, 1]$ and $|P| = \max_k(t_k - t_{k-1})$ its norm. The length $L(\gamma)$ of a (continuous) path $\gamma: [0, 1] \rightarrow X$ is defined by

$$(1) \quad L(\gamma) = \sup_P \sum_{k=1}^n d(\gamma(t_{k-1}), \gamma(t_k)) = \lim_{|P| \rightarrow 0} \sum_{k=1}^n d(\gamma(t_{k-1}), \gamma(t_k)),$$

where the sup is taken over all finite partitions of $[0, 1]$ and the second equality is easily proved. The length metric $\bar{d}: X \times X \rightarrow [0, \infty]$ induced by d is given by

$$\bar{d}(p, q) = \inf \{L(\gamma) : \gamma: [0, 1] \rightarrow X \text{ is a path in } (X, d) \text{ joining } p \text{ to } q\} \quad (p, q \in X).$$

Thus $\bar{d}(p, q) = \infty$ if and only if no d -rectifiable path connecting p and q exists.

For $n \in \mathbb{N}^+$, the set $\{1, \dots, n\}$ will be denoted by $[n]$. Given $\varepsilon > 0$, an ε -chain joining p to q is a string $c = (x_0, x_1, \dots, x_n)$ of points in X satisfying $x_0 = p$, $x_n = q$ and $d(x_{k-1}, x_k) \leq \varepsilon$ for all $k \in [n]$. Its length $L(c)$ equals $\sum_{k=1}^n d(x_{k-1}, x_k)$. For $p, q \in X$, define

$$d_\varepsilon(p, q) = \inf \{L(c) : c \text{ is an } \varepsilon\text{-chain joining } p \text{ to } q\}.$$

It is easily seen that $d_\varepsilon: X \times X \rightarrow [0, \infty]$ satisfies all the axioms for a metric for any $\varepsilon > 0$. Notice that $d_\varepsilon(p, q) = d(p, q)$ if the latter is not greater than ε . Finally, $d_0: X \times X \rightarrow [0, \infty]$ is defined by

$$d_0(p, q) = \sup_{\varepsilon > 0} d_\varepsilon(p, q) = \lim_{\varepsilon \rightarrow 0} d_\varepsilon(p, q) \quad (p, q \in X),$$

where the second equality follows from the fact that $d_{\varepsilon'} \geq d_\varepsilon$ if $\varepsilon' \leq \varepsilon$. The proof that d_0 is indeed a metric is left to the reader. It may be regarded as a discrete or discontinuous version of the length metric \bar{d} . The former is defined as the supremum of infima, and the latter as the infimum of suprema. The purpose of this note is to study d_0 , especially in its relation to d and \bar{d} .

Summary of results. The main result states that d_0 agrees with \bar{d} provided that one of the following conditions is satisfied: (i) (X, d) is complete and locally compact; (ii) (X, d) is complete and $(d_0)_0 = d_0$; (iii) (X, d) is a length space (that is, $\bar{d} = d$). Moreover, if (i) holds then any $p, q \in X$ such that $d_0(p, q) < \infty$ can be joined by a path whose d_0 -length equals $d_0(p, q)$. This theorem is proved in §1. Examples showing that none of its hypotheses can be omitted are constructed in §2. In particular, we exhibit a space (X, d) for which $d_0 \neq \bar{d}$ even though (X, d) is complete, σ -compact (hence separable), path-connected and locally path-connected, both through rectifiable paths.

Given metrics ρ, ρ' on X , let us write $\rho \leq \rho'$ to mean that $\rho(p, q) \leq \rho'(p, q)$ for all $p, q \in X$, and $\rho < \rho'$ when moreover strict inequality holds for at least one pair of points. In §3 we study the iterates of d_0 , i.e., the metrics d_0^n inductively defined by $d_0^{n+1} = (d_0^n)_0$, where $d_0^0 = d$. It is proved

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that if (X, d) is complete and $d_0^n = d_0^{n+1}$ for some $n \in \mathbf{N}$, then $d_0^m = \bar{d}$ for all $m \geq n$. We construct complete spaces (Y_n, d) where

$$d < d_0 < \dots < d_0^n < d_0^{n+1} = \bar{d},$$

and a complete space (Y_∞, d) for which $d_0^n < d_0^{n+1}$ for all $n \in \mathbf{N}$, but $\lim_n d_0^n < \bar{d}$. This should be compared to the behavior of the $\bar{\cdot}$ operation, which is always idempotent.

1. GENERAL RESULTS

(1.1) Lemma. *Let (X, d) be any metric space. Then*

$$(2) \quad d \leq d_0 \leq \bar{d}.$$

Proof. It is clear that $d \leq d_0$ since $d \leq d_\varepsilon$ for all $\varepsilon > 0$. Given a path γ joining p to q , a sum as in (1) coincides with the length of a corresponding ε -chain $(\gamma(0), \gamma(t_1), \dots, \gamma(1))$ as soon as $|P| \leq \varepsilon$. Thus $d_\varepsilon(p, q) \leq L(\gamma)$ for any such path γ , whence $d_\varepsilon \leq \bar{d}$ for all $\varepsilon > 0$ and $d_0 \leq \bar{d}$. \square

(1.2) Lemma. *If (X, d) is complete, then so are (X, d_0) and (X, \bar{d}) .*

Proof. Suppose first that $(x_n)_{n \in \mathbf{N}}$ is a d_0 -Cauchy sequence. Then it is also d -Cauchy by (2), hence it d -converges to some $x \in X$ by hypothesis. Let $\varepsilon > 0$ be given. Take $n_0 \in \mathbf{N}$ such that $i, j \geq n_0$ implies $d_0(x_i, x_j) \leq \varepsilon$. Given $\delta > 0$, choose $m \geq n_0$ so that $d(x_m, x) \leq \min\{\delta, \varepsilon\}$. Then $d_\delta(x_m, x) = d(x_m, x)$, hence

$$d_\delta(x_n, x) \leq d_\delta(x_n, x_m) + d_\delta(x_m, x) \leq d_0(x_n, x_m) + d(x_m, x) \leq 2\varepsilon \text{ for all } n \geq n_0.$$

Since δ is arbitrary, it follows that $d_0(x_n, x) \leq 2\varepsilon$ for all $n \geq n_0$. Therefore (x_n) d_0 -converges to x .

Suppose now that $(x_n)_{n \in \mathbf{N}}$ is a \bar{d} -Cauchy sequence. Then, as above, it must have a d -limit x . Choose $n_1 \leq n_2 \leq \dots$ such that $\bar{d}(x_i, x_j) < 2^{-\nu}$ for all $i, j \geq n_\nu$. Let $\varepsilon > 0$ be given, take $\nu_0 \in \mathbf{N}$ satisfying $2^{-\nu_0+1} \leq \varepsilon$ and let $n \geq n_{\nu_0}$. Choose a path $\gamma_0: [0, \frac{1}{2}] \rightarrow X$ joining x_n to $x_{n_{\nu_0+1}}$ of length less than $2^{-\nu_0}$. For each $k \geq 1$, let $\gamma_k: [1-2^{-k}, 1-2^{-k-1}] \rightarrow X$ be a path of length less than $2^{-\nu_0-k}$ joining $x_{n_{\nu_0+k}}$ to $x_{n_{\nu_0+k+1}}$. Finally, define $\gamma: [0, 1] \rightarrow X$ to be the concatenation of all the γ_k . To be precise, set

$$\gamma(t) = \gamma_k(t) \text{ if } t \in [1-2^{-k}, 1-2^{-k-1}] \text{ (} k \geq 0 \text{) and } \gamma(1) = x.$$

Since (x_n) d -converges to x , γ is indeed d -continuous at $t = 1$. Further, its length is at most $2^{-\nu_0+1} \leq \varepsilon$. Therefore $\bar{d}(x_n, x) \leq \varepsilon$ for all $n \geq n_{\nu_0}$ and (x_n) \bar{d} -converges to x . \square

(1.3) Example. Let (X, d) be the subspace of \mathbf{R}^2 (with the Euclidean metric) which consists of $[0, 1] \times \{0\}$ and the vertical segments of length 1 based at $(0, 0)$ and $(\frac{1}{k}, 0)$ for each $k \in \mathbf{N}^+$. Then (X, d) is compact and $d_0 = \bar{d}$, but (X, d_0) is not locally compact.

(1.4) Remark. Let $\rho \leq \rho'$ be two metrics on X . Then $\bar{\rho} \leq \bar{\rho}'$ and $\rho_0 \leq \rho'_0$. Indeed, if $\gamma: [0, 1] \rightarrow X$ is ρ' -continuous, then it is ρ -continuous and $L_\rho(\gamma) \leq L_{\rho'}(\gamma)$. Similarly, an ε -chain for ρ' is also an ε -chain for ρ , and its ρ -length is smaller than its ρ' -length.

(1.5) Lemma. *Let (X, d) be a metric space. Then d_0 is lower semicontinuous with respect to d :*

$$\liminf_{p_n \xrightarrow{d} p, q_n \xrightarrow{d} q} d_0(p_n, q_n) \geq d_0(p, q) \text{ for all } p, q \in X.$$

Proof. For any $p, q \in X$ and $\varepsilon > 0$,

$$\liminf_{p_n \xrightarrow{d} p, q_n \xrightarrow{d} q} d_0(p_n, q_n) \geq \liminf_{p_n \xrightarrow{d} p, q_n \xrightarrow{d} q} d_\varepsilon(p_n, q_n) = d_\varepsilon(p, q),$$

where the equality comes from the fact that d_ε agrees with d at distances smaller than ε . Letting $\varepsilon \rightarrow 0$ the desired inequality is obtained. \square

Remark. The length metric \bar{d} is generally not semicontinuous from either side with respect to d , as can be shown by means of simple examples. Although d_ε is continuous with respect to d for all $\varepsilon > 0$, d_0 itself need not be continuous, as illustrated by $X = \{0\} \cup \{\frac{1}{k} : k \in \mathbf{N}^+\} \subset \mathbf{R}$.

(1.6) Theorem. *Let (X, d) be a metric space. Suppose that one of the following holds:*

- (i) (X, d) is complete and locally compact.

- (ii) (X, d) is complete and $(d_0)_0 = d_0$.
- (iii) (X, d) is a length space (that is, $\bar{d} = d$).

Then $d_0 = \bar{d}$. Moreover, if (i) holds then (X, d_0) is a geodesic space.

The latter assertion means that for any $p, q \in X$ for which $d_0(p, q) < \infty$, there exists a d_0 -continuous path $\gamma: [0, 1] \rightarrow X$ joining p to q whose d_0 -length is $d_0(p, q)$. Note that condition (i) is satisfied if (X, d) is proper (i.e., if any d -ball is precompact) and in particular if (X, d) is compact. In cases (ii) and (iii), it cannot be guaranteed that $(X, d_0) = (X, \bar{d})$ is a geodesic space; e.g., let X be the metric graph consisting of two vertices and one edge of length $1 + \frac{1}{k}$ connecting them for each $k \in \mathbf{N}^+$.

(1.7) Corollary. *Let (X, d) be a metric space satisfying any of conditions (i)–(iii). Then*

$$d_0 = \bar{d} = (d_0)_0 = \overline{d_0} = \bar{d}_0.$$

Proof. Immediate from (2), (1.6) and the fact that $\bar{\bar{d}}$ always coincides with \bar{d} (a proof of the latter can be found in [1], pp. 32–33 or [2], pp. 37–38). \square

The remainder of this section is dedicated to the proof of the theorem.

(1.8) Lemma. *Let c_n be δ_n -chains joining $x, y \in (X, d)$, with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ ($n \in \mathbf{N}$). Then*

$$d_0(x, y) \leq \liminf L(c_n).$$

Proof. Immediate from the relations

$$d_0(x, y) = \lim_n d_{\delta_n}(x, y) \text{ and } d_{\delta_n}(x, y) \leq L(c_n). \quad \square$$

(1.9) Lemma. *Let (X, d) be a metric space and $p, q \in X$, $d_0(p, q) < \infty$. For each $n \in \mathbf{N}$, let $c_n = (x_0, \dots, x_{N_n})$ be an ε_n -chain joining p to q , with $L(c_n) \rightarrow d_0(p, q)$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Assume the existence of $k_n \in [N_n]$ so that $(x_{k_n})_{n \in \mathbf{N}}$ d -converges to some $x \in X$. Then $d_0(p, q) = d_0(p, x) + d_0(x, q)$,*

$$(3) \quad \sum_{k=1}^{k_n} d(x_{k-1}, x_k) \rightarrow d_0(p, x) \text{ and } \sum_{k=k_n+1}^{N_n} d(x_{k-1}, x_k) \rightarrow d_0(x, q) \text{ as } n \rightarrow \infty.$$

Proof. Set $\delta_n = \max\{\varepsilon_n, d(x_{k_n}, x)\}$. Then for each $n \in \mathbf{N}$, (x_0, \dots, x_{k_n}, x) and $(x, x_{k_n}, \dots, x_{N_n})$ are δ_n -chains joining p to x and x to q , respectively. Furthermore, if

$$s_n = \sum_{k=1}^{k_n} d(x_{k-1}, x_k) + d(x_{k_n}, x) \text{ and } t_n = d(x, x_{k_n}) + \sum_{k=k_n+1}^{N_n} d(x_{k-1}, x_k),$$

then $(s_n + t_n) \rightarrow d_0(p, q)$ by hypothesis. By (1.8), $d_0(p, x) \leq \liminf s_n$ and $d_0(x, q) \leq \liminf t_n$. On the other hand,

$$\limsup s_n + \liminf t_n \leq \lim(s_n + t_n) = d_0(p, q) \leq d_0(p, x) + d_0(x, q),$$

whence $\limsup s_n \leq d_0(p, x)$. Thus $\lim s_n$ exists and equals $d_0(p, x)$; similarly, $\lim t_n = d_0(x, q)$. Consequently $d_0(p, q) = d_0(p, x) + d_0(x, q)$. \square

In what follows the open ball centered at p of radius r with respect to a metric ρ is denoted by $B_\rho(p; r)$. If $S \subset X$, we denote by $B_\rho(S; r)$ the union of all balls $B_\rho(p; r)$ with $p \in S$. Also, $[t]$ denotes the greatest integer smaller than or equal to $t \in \mathbf{R}$. The main step in the proof of (1.6) is the following weak additivity property for d_0 .

(1.10) Lemma. *Let (X, d) be a locally compact metric space. Suppose that $B_{d_0}(p; r)$ is d -precompact and $r \leq d_0(p, q) < \infty$. Then for all sufficiently small $\delta > 0$, there exist $p_0 = p, \dots, p_N \in X$ ($N = [r/\delta]$) such that:*

$$(4) \quad d_0(p, q) = d_0(p_0, p_1) + \dots + d_0(p_{N-1}, p_N) + d_0(p_N, q) \text{ and } d_0(p_{k-1}, p_k) = \delta \text{ for all } k \in [N].$$

Proof. Using local compactness of (X, d) , choose for each $y \in \overline{B_{d_0}(p; r)}$ an $\varepsilon_y > 0$ such that $B_d(y; 2\varepsilon_y)$ is d -precompact. Now extract a finite subcover of $\overline{B_{d_0}(p; r)}$ by finitely many $B_d(y_j; \varepsilon_{y_j})$ and set $\varepsilon = \min\{\varepsilon_{y_j}\}$. It is easily checked that $B_d(x; \varepsilon)$ is d -precompact for any $x \in B_{d_0}(p; r)$.

Let $\delta \in (0, \varepsilon)$. For each $n \in \mathbf{N}^+$, choose a $\frac{1}{n}$ -chain (x_0, \dots, x_{N_n}) connecting p to q of length less than $d_{\frac{1}{n}}(p, q) + \frac{1}{n}$, and let k_n be the greatest element of $[N_n]$ satisfying

$$\sum_{k=1}^{k_n} d(x_{k-1}, x_k) < \delta.$$

Then $x_{k_n} \in B_d(p; \delta)$ for all n by the triangle inequality. Since this ball is d -precompact, passing to a subsequence if necessary, it can be assumed that (x_{k_n}) d -converges to some $p_1 \in X$ as $n \rightarrow \infty$. From (1.9) it follows that $d_0(p, q) = d_0(p, p_1) + d_0(p_1, q)$ and

$$d_0(p, p_1) = \lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} d(x_{k-1}, x_k) = \delta$$

by the choice of k_n . To obtain p_2 , apply the same procedure with p_1 in place of p , and so on inductively. \square

(1.11) Corollary. *Let (X, d) be a locally compact metric space. Suppose that $B_{d_0}(p; r)$ is d -precompact and $d_0(p, q) \geq r$. Then*

$$d(q, B_{d_0}(p; r)) \leq d_0(q, B_{d_0}(p; r)) = d_0(p, q) - r.$$

Proof. Immediate from (1.10) and the inequality $d \leq d_0$. \square

(1.12) Lemma. *If (X, d) is locally compact and complete, then any d_0 -ball is d -precompact.*

Proof. Let $p \in X$ be arbitrary. By local compactness of (X, d) , there exists some $\delta > 0$ such that $B_d(p; \delta)$ is d -precompact. Since $B_{d_0}(p; \delta) \subset B_d(p; \delta)$, we have that

$$R := \sup \{r \in \mathbf{R} : B_{d_0}(p; r) \text{ is } d\text{-precompact}\} \geq \delta.$$

Suppose for the sake of obtaining a contradiction that R is finite. Let $\varepsilon \in (0, R)$ be arbitrary. Using the fact that $B_{d_0}(p; R - \varepsilon)$ is d -totally bounded, cover the latter by finitely many balls $B_d(p_j; \varepsilon)$ ($j \in [m]$), where $p_j \in B_{d_0}(p; R - \varepsilon)$ for each j . Then, by (1.11),

$$B_{d_0}(p; R) \subset \bigcup_{j=1}^m B_d(p_j; 2\varepsilon).$$

Therefore $B_{d_0}(p; R)$ is d -totally bounded. Its d -closure is also totally bounded and in addition complete, as a closed subset of the complete space (X, d) . Thus $B_{d_0}(p; R)$ is d -precompact. In particular, it can be covered by finitely many balls $B_d(x_i; \delta_i)$ ($i \in [l]$) such that each $B_d(x_i; 2\delta_i)$ is d -precompact. Again by (1.11), if $\delta = \min_i \{\delta_i\}$ then

$$B_{d_0}(p; R + \delta) \subset \bigcup_{i=1}^l B_d(x_i; 2\delta_i).$$

Hence the former is d -precompact, contradicting the choice of R . \square

Remark. It need not be true that the d_0 -balls are d_0 -precompact even if (X, d) is compact, as shown by (1.3).

Proof of (1.6). Suppose that hypothesis (i) of the theorem holds and let $p, q \in X$ be arbitrary. If $d_0(p, q)$ is not finite, then by (2) neither is $\bar{d}(p, q)$, hence in this case they coincide. Assume then that $r = d_0(p, q)$ is finite. The ball $B_{d_0}(p; r)$ is precompact by (1.12). Applying (1.10) one deduces that for all sufficiently large $n \in \mathbf{N}$, say $n \geq n_0$, it is possible to find $p = p_0^n, \dots, p_{N_n+1}^n = q$ for which (4) holds with $\delta = \frac{1}{n}$, where $N_n = \lfloor rn \rfloor$. Let $S \subset [0, 1]$ be a countable dense subset. For each $n \geq n_0$, define $\gamma_n : S \rightarrow X$ by $\gamma_n(s) = p_{\lfloor N_n s \rfloor}^n$ ($s \in S$). Then γ_n is discontinuous, but

$$(5) \quad d(\gamma_n(s), \gamma_n(s')) \leq d_0(\gamma_n(s), \gamma_n(s')) \leq \frac{1}{n} |\lfloor N_n s \rfloor - \lfloor N_n s' \rfloor| \leq r |s - s'| + \frac{2}{n} \quad (s, s' \in S).$$

The triangle inequality for d_0 shows that p_k^n lies in the compact set $\overline{B_{d_0}(p; r)}$ for all $k \in [N_n]$. Using Cantor's diagonal argument one can obtain a subsequence (γ_{n_ν}) with the property that for all $s \in S$, $\gamma_{n_\nu}(s)$ d -converges to some $\gamma(s) \in X$ as $\nu \rightarrow \infty$. Then, by (5),

$$d(\gamma(s), \gamma(s')) \leq r |s - s'| \text{ for all } s, s' \in S.$$

Since (X, d) is complete, γ can be continuously extended to $[0, 1]$, so that

$$d(\gamma(t), \gamma(t')) \leq r |t - t'| \text{ for all } t, t' \in [0, 1].$$

This implies that $L(\gamma) \leq d_0(p, q) = r$. But $d_0 \leq \bar{d}$ and $\bar{d}(p, q) \leq L(\gamma)$ by the definition of \bar{d} , hence

$$d_0(p, q) = \bar{d}(p, q) = L(\gamma).$$

Thus $(X, \bar{d}) = (X, d_0)$ is a geodesic metric space.

Now assume that (ii) is satisfied. Since $d_0 \leq (d_0)_\varepsilon \leq (d_0)_0$ always holds, we deduce that $(d_0)_\varepsilon = d_0$ for all $\varepsilon > 0$. Therefore, given $p, q \in X$ with $d_0(p, q) < \infty$ and $\varepsilon > 0$, one can find an ε -chain c (with respect to d_0) joining p to q for which $L_{d_0}(c) \leq d_0(p, q) + \varepsilon$. Hence (X, d_0) admits “approximate midpoints” (cf. [1], p. 32 or [2], p. 42). Since (X, d_0) is complete by (1.2), it must be a length space, i.e., $\bar{d}_0 = d_0$. On the other hand, (2) and (1.4) imply that

$$d \leq d_0 \leq \bar{d} \leq \bar{d}_0.$$

Therefore $d_0 = \bar{d}$.

Finally, if condition (iii) holds then it is obvious from (2) that $d_0 = \bar{d}$. □

2. EXAMPLES

It is erroneously asserted in Exercise 3.1.26 of [2] that d_0 coincides with \bar{d} whenever (X, d) is complete.[†] However, as the following examples show, none of the hypotheses in (i) and (ii) of (1.6) can be omitted.

(2.1) *Example* (cf. Figure 1). For each integer $k \geq 2$, let

$$\begin{aligned} S_k = & \{(1, 0, \dots, 0, t, 0, \dots) \in \ell^\infty : t \in [0, 1]\} \cup \\ & \cup \{(t, 0, \dots, 0, 1, 0, \dots) \in \ell^\infty : t \in [\frac{1}{k}, 1]\} \cup \\ & \cup \{(\frac{1}{k}, 0, \dots, 0, t, 0, \dots) \in \ell^\infty : t \in [0, 1]\}, \end{aligned}$$

where in each description the only nonzero coordinates are the first and k -th. Let

$$p = (0, 0, \dots), \quad q = (1, 0, 0, \dots)$$

and let $Y = \{p\} \cup \bigcup_{k=2}^\infty S_k$, equipped with the metric d derived from $\|\cdot\|_\infty$. Then (Y, d) is complete, σ -compact (hence separable) and connected (though not path-connected). It is not locally compact, since no neighborhood of p or q is precompact. It will be proved shortly that $d_0(p, y) = 3 + \bar{d}(q, y)$ for all $y \neq p$. In particular,

$$d(p, q) = 1 < d_0(p, q) = 3 < \bar{d}(p, q) = \infty,$$

and q is simultaneously the point d_0 -closest to p and one of the points maximizing the d -distance to p . Note that (Y, d_0) is disconnected even though (Y, d) is connected, and that the d_0 -topology is strictly finer than the d -topology, even though d_0 only takes on finite values.

In the sequel we say that an ε -chain c can be *reduced* to another ε -chain c' if they join the same pair of points and $L(c') \leq L(c)$.

(2.2) Lemma. *Let (Y, d) be as described in (2.1). Then $d_0(p, y) = 3 + \bar{d}(q, y)$ for any $y \neq p$, and the restrictions of d_0 and \bar{d} to $Y \setminus \{p\}$ coincide. Consequently, $d < d_0 < (d_0)_0 = \bar{d}$.*

[†]The definition of d_ε (and hence that of d_0) in [2] is different from the one we have given, but the two are easily proved to be equivalent.

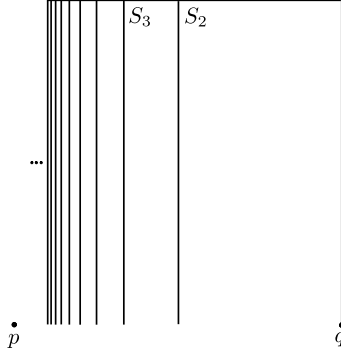


FIGURE 1. A two-dimensional representation of the space Y described in (2.3). The segments are drawn to scale, but the figure is highly distorted since the S_k lie in mutually distinct planes of ℓ^∞ and $S_k \cap S_l = \{q\}$ for $k \neq l$.

Proof. Suppose that $x \in S_m$, $y \in S_l$. Take $\varepsilon < \min\{\frac{1}{m(m+1)}, \frac{1}{l(l+1)}\}$. Using that

$$B_d(S_m \cup S_l; \varepsilon) = S_m \cup S_l \cup B_d(q; \varepsilon),$$

it is not hard to see that any ε -chain joining x to y can be reduced to a chain contained in $S_m \cup S_l$. Any such chain has length at least $\bar{d}(x, y) - 4\varepsilon$, as one verifies directly. Thus $d_0(x, y) \geq \bar{d}(x, y)$, and the reverse inequality is guaranteed by (2).

Now let $y \in S_l$ and $\varepsilon < \frac{1}{l(l+1)}$. Since $B_d(S_l; \varepsilon) = S_l \cup B_d(q; \varepsilon)$, any ε -chain joining y to p can be reduced to the concatenation of chains c_1 and c_2 joining y to q and q to p , respectively. Let $x = (\frac{1}{m}, 0, \dots)$, where m is the largest integer satisfying $\frac{1}{m} \leq \varepsilon$. Adjoining x to c_2 , an ε -chain c joining q to x satisfying $L(c) \leq L(c_2) + \varepsilon$ is obtained. But direct verification shows that $L(c) \geq \bar{d}(q, x) - 2\varepsilon$. Thus

$$L(c_2) \geq 3 - 3\varepsilon, \text{ while } L(c_1) \geq \bar{d}(q, y) - 4\varepsilon$$

as in the preceding paragraph. Hence $d_0(p, y) \geq 3 + \bar{d}(q, y)$. On the other hand, given $\varepsilon > 0$, an ε -chain (x_0, \dots, x_n) joining p to y of length at most $3 + \bar{d}(q, y)$ can easily be constructed: take $x_1 = (\frac{1}{m}, 0, \dots)$ with $m > \varepsilon^{-1}$ and choose the remaining x_k monotonically along the geodesic joining x_1 to y . Thus $d_0(p, y) = 3 + \bar{d}(q, y)$. This completes the proof of the asserted description of d_0 .

Since $d_0(p, y) \geq 3$ for all $y \neq p$, $(d_0)_0(p, y) = \infty = \bar{d}(p, y)$ for any such y . In addition, $(d_0)_0(x, y) = \bar{d}(x, y)$ for $x, y \neq p$ since $d_0(x, y) = \bar{d}(x, y)$ and $d_0 \leq (d_0)_0 \leq \bar{d}$ always holds. Therefore $d < d_0 < (d_0)_0 = \bar{d}$ as claimed. \square

It is natural to think that the failure of d_0 to agree with \bar{d} in the previous example has more to do with the fact that (Y, d) is not path-connected than with absence of local compactness; after all, \bar{d} is defined in terms of lengths of paths. However, this is not the case.

(2.3) *Example.* For each integer $k \geq 2$, let $f_k: [\frac{1}{k+1}, \frac{1}{k}] \rightarrow \mathbf{R}$ be a smooth function such that

$$f_k(\frac{1}{k+1}) = \frac{1}{k+1}, \quad f_k(\frac{1}{k}) = 0 \quad \text{and} \quad |f_k(t)| \leq t \quad \text{for all } t \in [\frac{1}{k+1}, \frac{1}{k}].$$

Let $\eta_k: [\frac{1}{k+1}, \frac{1}{k}] \rightarrow \ell^\infty$ be given by $\eta_k(t) = (t, 0, \dots, 0, f_k(t), 0, \dots)$, where f_k appears in the k -th coordinate. Finally, let $\gamma_k: [0, \frac{1}{k}] \rightarrow \ell^\infty$ be the concatenation of the line segment joining $p = (0, 0, \dots)$ to $\eta_k(\frac{1}{k+1})$ and η_k . Then γ_k joins p to $(\frac{1}{k}, 0, \dots)$ without passing through $(\frac{1}{m}, 0, \dots)$ for any $m \neq k$, and (because $|f_k(t)| \leq t$) the intersection of its image with any d -ball centered at p is an arc of γ_k . Let f_k be chosen so as to have $L(\gamma_k) = 3 + \frac{1}{k}$.

Now define X to be the union of the set Y of (2.1) and the images of all paths γ_k ($k \geq 2$), equipped with the restriction d of the metric of ℓ^∞ . Then (X, d) is not locally compact, but it is complete, σ -compact, and any pair of points in it can be joined by a rectifiable path. Moreover, given $x \in X$ and a d -neighborhood U of x , there exists a d -ball $B \subset U$ containing x such that any two points of B can be joined by a rectifiable path contained in B . Still,

$$d(p, q) = 1 < d_0(p, q) = 3 < \bar{d}(p, q) = 6.$$

A minimizing geodesic connecting p and q is the concatenation of γ_k and S_k , for any k .

(2.4) *Example.* Let $X = \mathbf{R}^2 \setminus (\{0\} \times [-1, +1])$, furnished with the restriction d of the Euclidean metric on \mathbf{R}^2 . Then $d_0 = d$, but if $p = (-1, 0)$, $q = (1, 0)$, then $d_0(p, q) = 2 < 2\sqrt{2} = \bar{d}(p, q)$. The space (X, d) is locally compact and (locally) path-connected through rectifiable paths, but not complete.

(2.5) *Example.* Let X be the set of all rational numbers, equipped with the restriction of the Euclidean metric d . Then $(d_0)_0 = d_0 = d$ does not coincide with \bar{d} .

3. ITERATES OF d_0

One of the fundamental properties of the induced length metric \bar{d} is that $\bar{\bar{d}} = \bar{d}$. In contrast, d_0 may not coincide with $(d_0)_0$, as shown by the space (Y, d) of (2.1). For a metric d and $n \in \mathbf{N}^+$, let

$$d_0^0 = d \text{ and } d_0^n = (d_0^{n-1})_0.$$

It follows from (2) and (1.7) (applied to (X, \bar{d})) that

$$d \leq d_0^{n-1} \leq d_0^n \leq \bar{d} \text{ for any } n \in \mathbf{N}^+.$$

(3.1) Proposition. *Let (X, d) be a complete metric space and suppose that $d_0^n = d_0^{n+1}$ for some $n \in \mathbf{N}$. Then $d_0^m = \bar{d}$ for all $m \geq n$.*

Proof. If $d_0 = d$ then certainly $(d_0)_0 = d_0$, hence it may be assumed that $n \geq 1$. As a consequence of (1.4) and the preceding inequalities,

$$\bar{d} \leq \overline{d_0^{n-1}}.$$

The metric d_0^{n-1} is complete by repeated use of (1.2). By case (ii) of (1.6) applied to d_0^{n-1} ,

$$d_0^n = (d_0^{n-1})_0 = \overline{d_0^{n-1}}.$$

Therefore $\bar{d} = d_0^n = d_0^{n+1} = \dots$. □

Remark. The assumption that (X, d) is complete cannot be omitted, as shown by (2.5).

We shall now describe complete connected spaces (Y_n, d) for which

$$d < d_0 < \dots < d_0^n < d_0^{n+1} = \bar{d},$$

and a complete connected space (Y_∞, d) for which $\lim_{n \rightarrow \infty} d_0^n < \bar{d}$.

(3.2) *Example.* Set $(Y_1, d) = (Y, d)$, where the latter was described in (2.1).

For each $m \in \mathbf{N}^+$, take the disjoint union of m copies Z_m^j ($j \in [m]$) of Y_1 with its metric d contracted by the factor $\frac{1}{m}$. Let Y_2 be the result of gluing:

- (a) The point q of Z_m^j to the point p of Z_m^{j+1} for each $j \in [m-1]$ and $m \in \mathbf{N}^+$.
- (b) The points p of Z_m^1 for all $m \in \mathbf{N}^+$ to a single point, still denoted p .
- (c) The points q of Z_m^m for all $m \in \mathbf{N}^+$ to a single point, still denoted q .

The spaces Y_n for integer $n \geq 2$ are defined inductively by replacing Y_1 and Y_2 by Y_{n-1} and Y_n in the preceding paragraph, respectively. Note that Y_n contains an isometric copy of Y_{n-1} , namely, Z_1^1 (or more precisely its image under the gluing), and the points p and q of Y_{n-1} are thereby identified with the corresponding points p and q of Y_n . Finally, let $Y_\infty = \bigcup_{n=1}^\infty Y_n$, where Y_{n-1} is regarded as a subspace of Y_n as was just indicated. The metric in Y_∞ is uniquely determined since any two points in it lie in the same Y_n whenever n is sufficiently large.

Straightforward inductive arguments show that Y_n is connected and complete for every n ; Y_∞ is connected as the union of an increasing family of connected spaces, and it is complete because a Cauchy sequence either eventually lies in some Y_n , or converges to p or q .

(3.3) Proposition. *Let (Y_n, d) be as in (3.2). Then $d_0^n(p, y) = 3 + \bar{d}(q, y)$ for all $y \neq p$ and the restrictions of d_0^n and \bar{d} to $Y_n \setminus \{p\}$ agree. In particular,*

$$(6) \quad d < d_0 < \dots < d_0^{n-1} < d_0^n < d_0^{n+1} = \bar{d} \text{ in } Y_n.$$

In (Y_∞, d) we have that $d_0^n < d_0^{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n \rightarrow \infty} d_0^n < \bar{d}$.

Proof. By induction on n . For $n = 1$ the assertions were proved in (2.2). Assume that the conclusion holds for Y_{n-1} with $n \geq 2$. From now on Z_m^j denotes the image of this space under the gluing which defines Y_n . Recall that Z_m^j is isometric to $(Y_{n-1}, \frac{1}{m}d_{Y_{n-1}})$ for all $j \in [m]$, and note that the restriction of \bar{d} to Z_m^j and the length metric on the latter induced by the restriction of d coincide.

Let $j < m$, $p \neq x \in Z_m^j$ and $y \in Y_n$ be arbitrary. If $y' \notin Z_m^j$, then $d_0^{n-1}(x, y') \geq \frac{3}{m}$ (here the hypotheses that $j < m$ and $x \neq p$ are essential). Hence

$$d_0^n(x, y) = \bar{d}(x, y) = \infty \text{ for all } y \notin Z_m^j.$$

If $y \in Z_m^j$, then $d_0^n(x, y) = \bar{d}(x, y)$ by the induction hypothesis applied to Z_m^j . In particular, taking $y = p$,

$$(7) \quad d_0^n(p, x) = \bar{d}(p, x) = \infty = 3 + \bar{d}(q, p) \text{ for } p \neq x \in Z_m^j \text{ with } j < m.$$

Now let $x, y \in Z_m^m \setminus Z_m^{m-1}$, where we set $Z_1^0 = \{p\}$ for convenience, and let $\varepsilon < \frac{3}{m}$. Let $c = (x = x_0, \dots, x_N = y)$ be an ε -chain with respect to d_0^{m-1} and assume that c is not contained in Z_m^m . Let $k_0, k_1 \in [N]$ be the greatest (resp. smallest) indices such that $x_0, \dots, x_{k_0} \in Z_m^m$ and $x_{k_1}, \dots, x_N \in Z_m^m$. Then $d_0^{m-1}(x_{k_0}, q), d_0^{m-1}(x_{k_1}, q) \leq \varepsilon$ by the choice of ε and k_i (informally, the chain cannot leave Z_m^m through Z_m^{m-1} since $\varepsilon < \frac{3}{m}$). Replacing $(x_{k_0}, \dots, x_{k_1})$ by (x_{k_0}, q, x_{k_1}) , we obtain an ε -chain c' contained in Z_m^m satisfying $L(c') \leq L(c) + 2\varepsilon$. This shows that the restriction of d_0^n to Z_m^m agrees with $(d|_{Z_m^m \times Z_m^m})_0^n$, that is, with the original metric d_0^n on Z_m^m . Hence, by the induction hypothesis applied to Z_m^m ,

$$d_0^n(x, y) = \bar{d}(x, y) \text{ for } x, y \in Z_m^m \setminus Z_m^{m-1}, m \in \mathbf{N}^+.$$

Now let $x \in Z_m^m \setminus Z_m^{m-1}$ and $y \in Z_l^l \setminus Z_l^{l-1}$ with $m \neq l$, and take $\varepsilon < \min\{\frac{3}{m}, \frac{3}{l}\}$. A straightforward modification of the preceding argument can be used to compare an ε -chain joining x to y to the concatenation of ε -chains joining x to q and q to y , allowing one to conclude that

$$d_0^n(x, y) = \bar{d}(x, q) + \bar{d}(q, y) = \bar{d}(x, y).$$

It remains to compute $d_0^n(p, y)$ for $y \in Z_l^l \setminus Z_l^{l-1}$. Using the argument of the preceding paragraph again, one deduces that

$$d_0^n(p, y) = d_0^n(p, q) + d_0^n(q, y) = d_0^n(p, q) + \bar{d}(q, y).$$

Any ε -chain joining p to q can be reduced to a chain c' contained in $\bigcup_{j=1}^m Z_m^j$ for some $m > \varepsilon^{-1}$. By the definition of gluing and the inductive hypothesis, $d_0^{m-1}(a, b) \geq \frac{3}{m}|i - j|$ whenever $a \in Z_m^i$, $b \in Z_m^j$ ($a \neq b$). Therefore $L(c') \geq 3 - \frac{1}{m} \geq 3 - \varepsilon$, so that $d_0^n(p, q) \geq 3$. On the other hand, it is easy to construct explicit ε -chains of length 3 joining p to q for any $\varepsilon > 0$: take $m > \varepsilon^{-1}$ and consider $(p, q_1, \dots, q_{m-1}, q)$, where $q_j \in Z_m^j$ denotes the point glued to Z_m^{j+1} in step (a) of (3.2). Thus $d_0^n(p, y) = 3 + \bar{d}(q, y)$ if $y \in Z_m^m \setminus Z_m^{m-1}$ for some m , and for other $y \neq p$ this equality holds by (7).

This completes the description of the metric d_0^n on Y_n . Since $d_0^n(p, y) \geq 3$ for all $y \in Y_n \setminus \{p\}$,

$$d_0^{n+1}(p, y) = \infty = \bar{d}(p, y) \text{ for all such } y.$$

For $x, y \neq p$, certainly $d_0^{n+1}(x, y) = \bar{d}(x, y)$, since $d_0^n \leq d_0^{n+1} \leq \bar{d}$ always holds and $d_0^n(x, y) = \bar{d}(x, y)$ already. Hence $d_0^{n+1} = \bar{d}$ in Y_n . The inequalities in (6) are a consequence of (3.1) and the fact that $d_0^n < \bar{d}$.

Finally, since $d_0^n(p, q) = 3$ in Y_n and Y_n is isometrically embedded in Y_∞ for all $n \in \mathbf{N}^+$,

$$\lim_{n \rightarrow \infty} d_0^n(p, q) \leq 3 < \bar{d}(p, q) = \infty \text{ in } Y_\infty.$$

Thus $\lim_n d_0^n < \bar{d}$. Since (Y_∞, d) is complete, (3.1) guarantees that $d_0^n < d_0^{n+1}$ for all $n \in \mathbf{N}$. \square

Remark. By replacing Y by X throughout in (3.2), where (X, d) is the space described in (2.3), one obtains a family (X_n, d) of spaces satisfying the same inequalities as in (6), but with the additional property that each (X_n, d) is (locally) path-connected through rectifiable paths. To prove this one can use the argument given above, with small variations; however, the proof is more cumbersome since the metric d_0 on X_n does not have such a simple description as that on Y_n .

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